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The stability of stationary waves in a wavy-walled channel

By JOHN PHILIP McHUGH
Department of Mechanical Engineering, University of New Hampshire, Durham, NH 03824-3591, USA

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Experiments by Binnie showed that unsteady waves were produced by flow through a channel with symmetric, wavy sidewalls, with waves propagating both upstream and downstream. However, the first-order solution to this problem that was obtained by Yih is a set of steady waves. The steady solution is shown to be unstable to a pair of infinitesimal disturbance waves which satisfy the resonance conditions of Phillips. For the Froude-number range used by Binnie, a pair of disturbances has been found such that one wave propagates upstream, one propagates downstream, and the amplitudes have an exponential growth. The Froude numbers outside the range of Binnie are also shown to be unstable. The steady waves produced by flow through an antisymmetric channel are shown to be unstable in the same manner.

1. Introduction

The flow through a wavy-walled channel was first studied by Binnie (1960). His study consisted of a set of experiments where water flowed through several channels of different size with the intent of studying waves that result from the walls of the channel having a corrugated surface. The flow velocity, water depth, and wall corrugation were all varied during Binnie's investigation. The results of the experiments showed that a purely steady solution was never observed for any flow velocity, water depth, or wall corrugation. The resulting flow in each channel had waves that propagated both upstream and downstream, and the water surface was rather turbulent. Binnie measured the wavelengths and periods of the waves and noted that the measurement of wavelength was difficult due to an unsteady 'beating' motion of the free surface.

Yih (1982) studied the problem of Binnie (1960) analytically and found a steady solution for waves in a wavy-walled channel where the walls were symmetric about the centre of the channel, as they were in the experiments of Binnie (1960). The waves that resulted from the steady solution of Yih (1982) have a diamond pattern.

Yih (1983) also studied the waves that result when the walls of a wavy-walled channel are not symmetric about the centre of the channel but are antisymmetric (in-phase) with one another. The solution of Yih (1983) is a steady wave pattern, similar to the symmetric case. There are no published results of systematic experiments on waves created by water flowing through an antisymmetric, wavy-walled channel.

The demonstration of instability of basic waves requires disturbances which interact with the basic waves and grow with time. Phillips (1960) has shown that waves will interact only if there are (at least) three wavetrains present, and if the waves satisfy the following resonance conditions:

$$\sigma_1 = \sigma_2 + \sigma_0, \tag{1.1}$$

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_0. \tag{1.2}$$

The frequency of a particular wave train is σ and the wavenumber vector is $\boldsymbol{k}$. A complete discussion of these relations is given by Phillips (1977).

In Phillips' studies, three free wavetrains interact and energy flows among them. In this study, the primary waves are created by a stream flowing between the wavy walls, and energy flows from the primary waves to two disturbance wavetrains. The problem is therefore regarded as one of stability.

The stability of surface waves is a problem that has been studied extensively, and centres on the stability of the well-known Stokesian waves (see Stokes 1847), which are periodic, nonlinear, propagating waves in a semi-infinite fluid. Benjamin (1967) has shown that Stokesian waves are unstable to waves with sideband frequencies. Whitham (1967) has reached the same result using a variational method. Both Benjamin and Whitham showed that Stokesian waves are stable for small values of wave slope.

The present investigation follows the approach of Yih (1976), who showed that the waves created by flow over a wavy bottom at any Froude number are unstable to a pair of disturbance waves. The disturbance waves and the primary waves again must satisfy Phillips resonance conditions. Yih (1976) showed that the waves over a wavy bottom are unstable for any wavenumber of the bottom shape.

One important difference between Yih's investigation and those of Phillips, Benjamin, and Whitham is that Yih (1976) studied the stability of bound waves in a stream flowing at any Froude number, whereas the waves investigated by the others are free waves. Stronger instability is found for the bound waves using the approach of Yih (1976).

It will be demonstrated that the flow of a homogeneous fluid through a wavy-walled channel, with either symmetric walls or antisymmetric walls, is unstable to a pair of disturbance waves.

2. Basic equations

The problem under consideration is the flow of a liquid through a channel with wavy sides. The bottom of the channel is flat and the top is a free surface. Cartesian coordinates are chosen with the $\mathscr{x}$-axis parallel to the centreline of the channel, the $\mathscr{y}$-axis across the channel, and the $\mathscr{z}$-axis pointing vertically upward. The coordinate $\mathscr{z}$ is chosen for convenience to have a value of zero at the mean level of the free surface. A script letter indicates a dimensional quantity.

The average depth of the channel is h. The sides of the channel are straight in the $\mathscr{z}$-direction, and have a sinusoidal shape in the $\mathscr{x}$-direction. Two types of channel will be treated. One has opposite walls that are symmetric about the $\mathscr{y} = 0$ plane. This is the type of channel investigated experimentally by Binnie (1960). The shape of the wall for these symmetric walls is given by

$$\mathscr{y} = -L - a\,\sin k\mathscr{x}, \quad y = +L + a\,\sin k\mathscr{x},$$

where L is the half-width of the channel, k is the wavenumber of the side corrugation, and a is the amplitude of the side corrugation. The second wall shape considered has antisymmetric walls, such that one wall is in-phase with the opposite wall. Flow in

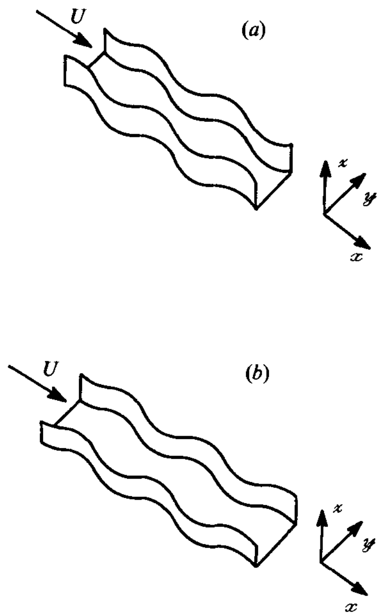

FIGURE 1. Channel shapes.

this type of channel was discussed analytically by Yih (1983). The wall shape for the antisymmetric walls is given by

$$y = +L + a \sin kx, \quad y = -L + a \sin kx.$$

The channel shape and the coordinate system for both channels are shown in figure 1.

The liquid flows in the positive x-direction with a speed of U, where U is defined as the average of the x-component of velocity across any (y, z)-plane. The velocity components in the x-, y- and z-directions are demoted by u, v and w, respectively. It is convenient at this point to non-dimensionalize the variables. Let the lengthscale be the mean half-width of the channel, L, and the velocity scale be the mean free-stream velocity, U. The non-dimensional variables are given by

$$u = \frac{u}{U}, \quad v = \frac{v}{U}, \quad w = \frac{w}{U}, \quad x = \frac{x}{L}, \quad y = \frac{y}{L}, \quad z = \frac{z}{L}.$$

The problem will be solved in completely non-dimensional variables.

The flow is assumed irrotational, and a velocity potential, ϕ, will be employed, in terms of which the velocity can be expressed:

$$u = \phi_x, \quad v = \phi_y, \quad w = \phi_z.$$

The subscripts x, y and z indicate partial differentiation. The velocity potential must satisfy Laplace's equation,

$$\phi_{zz} + \phi_{yy} + \phi_{zz} = 0, \tag{2.1}$$

along with the following solid boundary conditions:

$$\phi_z = 0 \quad \text{at } z = -d, \tag{2.2}$$

$$\phi_n = 0 \quad \text{on the sidewalls}, \tag{2.3}$$

where n is the distance normal to the sidewalls at any point and d is the dimensionless water depth given by

$$d = \frac{h}{L}. \tag{2.4}$$

The two boundary conditions that must be met at the free surface are the kinematic condition,

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z \quad \text{on } z = \eta, \tag{2.5}$$

and the dynamic condition,

$$\phi_t + \tfrac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] + \frac{1}{F^2}\eta = \text{constant} \quad \text{on } z = \eta, \tag{2.6}$$

where F is the Froude number,

$$F = \frac{U}{(gL)^{\frac{1}{2}}} \tag{2.7}$$

and η is the vertical elevation of the free-surface. Equations (2.1)–(2.6) define the boundary-value problem for flow through a wavy-walled channel.

3. Mean flows

Before discussing the steady solution to (2.1)–(2.6), the difficulty of meeting the boundary condition at the wavy sidewalls can be eliminated by using the coordinate transformation of Yih (1982). For symmetric wavy walls the transformation is given by

$$\left.\begin{aligned} x &= \alpha + a \sin k\alpha \cosh k\beta, \\ y &= \beta + a \cos k\alpha \sinh k\beta. \end{aligned}\right\} \tag{3.1}$$

The Jacobian is

$$J = \frac{\partial(x,y)}{\partial(\alpha,\beta)} = 1 + 2ak \cos k\alpha \cosh k\beta + a^2 k^2 \{\cos^2 k\alpha + \sinh^2 k\beta\}. \tag{3.2}$$

For antisymmetric channels, the transformation is

$$\left.\begin{aligned} y &= \alpha + a \sin k\alpha \sinh k\beta, \\ y &= \beta + a \cos k\alpha \cosh k\beta, \end{aligned}\right\} \tag{3.3}$$

and the Jacobian is

$$J = 1 + 2ak \cos k\alpha \sinh k\beta + a^2 k^2 (\sinh^2 k\beta + \sin^2 k\alpha). \tag{3.4}$$

The boundary of the channel for either channel shape is now at $\beta = \pm 1$, and the boundary condition at the sidewalls is

$$\phi_\beta = 0 \quad \text{at } \beta = \pm 1. \tag{3.5}$$

The problem can now be more easily solved in (α, β, z)-coordinates. Note that for $\beta = \pm 1$, (3.1) and (3.3) give only approximately the boundaries specified previously.

In terms of α, β and z, Laplace's equation becomes

$$J^{-1}(\phi_{\alpha\alpha} + \phi_{\beta\beta}) + \phi zz = 0. \tag{3.6}$$

The kinematic free-surface boundary condition, (2.5), becomes

$$\phi_t + J^{-1}(\phi_\alpha \eta_\alpha + \phi_\beta \eta_\beta) = \phi_z \quad \text{on } z = \eta, \tag{3.7}$$

and the dynamic free-surface condition, (2.6), is now

$$\phi_t + \frac{1}{2}\left[\frac{1}{J}(\phi_\alpha^2 + \phi_\beta^2) + \phi_z^2\right] + \frac{1}{F^2}\eta = \text{constant} \quad \text{on } z = \eta. \tag{3.8}$$

The condition at the bottom of the channel does not change, and is still given by (2.2).

Equations (3.7) and (3.8) are combined by eliminating η to get

$$\frac{1}{J}\left[\phi_\alpha\frac{\partial}{\partial\alpha}+\phi_\beta\frac{\partial}{\partial\beta}\right]\left[\frac{1}{J}(\phi_\alpha^2+\phi_\beta^2)+\phi_z^2\right]+\frac{2}{F^2}\phi_z=0, \tag{3.9}$$

which must be satisfied on $z=\eta$. The derivatives with respect to time have been set to zero in (3.9) for steady waves. Expand ϕ in a power series:

$$\phi=\phi_0+a\phi_1+a^2\phi_2+a^3\phi_3+\ldots, \tag{3.10}$$

where a is the amplitude of the side variation. Yih (1982) presents the following solution to order a for symmetric channels:

$$\phi=\alpha+a\sum_{n=0}^{\infty}B_n\sin k\alpha\cos n\pi\beta\cosh\gamma_n(z+d) \tag{3.11}$$

where γ_n is determined from Laplace's equation to be

$$\gamma_n=(k^2+n^2\pi^2)^{\frac{1}{2}}, \tag{3.12}$$

and B_n is given by

$$\left.\begin{aligned}B_nC_n\gamma_n^2&=2k^3(-1)^{n+1}\sinh k,\\ C_n&=-k^2\cosh\gamma_nd+F^{-2}\gamma_n\sinh\gamma_nd.\end{aligned}\right\} \tag{3.13}$$

The wave elevation to order a is

$$\eta=-ak^{-1}\cos k\alpha\sum_{n=0}^{\infty}B_n\gamma_n\sinh\gamma_nd\cos n\pi\beta. \tag{3.14}$$

For antisymmetric channels, the solution to order a given by Yih (1983) is

$$\phi=\alpha+a\sum_{n=1}^{\infty}B_n\sin k\alpha\sin(n-\tfrac{1}{2})\pi\beta\cosh\gamma_n(z+d), \tag{3.15}$$

where γ_n is determined from Laplace's equation to be

$$\gamma_n=(k^2+(n-\tfrac{1}{2})^2\pi^2)^{\frac{1}{2}}, \tag{3.16}$$

$$B_nC_n\gamma_n^2=2k^3(-1)^n\cosh k, \tag{3.17}$$

and C_n is as before. The free-surface elevation is given by

$$\eta=-ak^{-1}\cos k\alpha\sum_{n=1}^{\infty}B_n\gamma_n\sinh\gamma_nd\sin(n-\tfrac{1}{2})\pi\beta. \tag{3.18}$$

Equations (3.11) and (3.14) will be referred to as the mean flow for symmetric channels. Equations (3.15) and (3.18) will be referred to as the mean flow for antisymmetric channels. The waves described in this section will be called primary waves.

4. Stability equations

In order to show that the mean flows described above are unstable, a small disturbance will be added to the mean flow, and inserted into the equation of motion and boundary conditions. Before discussing the form of the disturbance, these equations will be developed in their general form. The transformation of coordinates that was used to find the mean flow is convenient also in showing the instability of the mean flow. Hence, the stability analysis will be conducted in (α, β, z)-coordinates.

Assume that the solution to (2.2), (3.5), (3.6), (3.7) and (3.8) is the sum of a mean flow and a disturbance:

$$\phi = \Phi + \tilde{\phi}, \quad \eta = \zeta + \tilde{\eta} \tag{4.1}$$

where Φ is the velocity potential for the mean flow, $\tilde{\phi}$ is the potential for a small disturbance, ζ is the surface elevation for the mean flow, and $\tilde{\eta}$ is the surface elevation for the disturbance. Note that the ϕ in (3.11) and (3.15) is now denoted by Φ, and η in (3.14) and (3.18) is now denoted by ζ. The stability problem will be formulated in terms of the mean and disturbance quantities only, hence the tildes on the disturbance quantities may be dropped. The disturbance quantities are henceforth represented by ϕ and η.

Substitution of (4.1) into Laplace's equation produces

$$J^{-1}(\phi_{\alpha\alpha} + \phi_{\beta\beta}) + \phi_{zz} = 0. \tag{4.2}$$

The bottom and sidewall boundary conditions on the disturbance quantities are

$$\phi_z = 0 \quad \text{on } z = -h, \tag{4.3}$$

$$\phi_\beta = 0 \quad \text{on } \beta = \pm 1. \tag{4.4}$$

Substituting (4.1) into the free-surface boundary conditions, one obtains

$$\phi_z = \eta_t + J^{-1}[\Phi_\alpha \eta_\alpha + \phi_\alpha \zeta_\alpha + \Phi_\beta \eta_\beta + \phi_\beta \zeta_\beta], \tag{4.5}$$

$$\phi_t + J^{-1}[\Phi_\alpha \phi_\alpha + \Phi_\beta \phi_\beta] + \Phi_z \phi_z + F^{-2}\eta = \text{constant}, \tag{4.6}$$

which must be satisfied on $z = \zeta + \eta$. Note that the products of the disturbance quantities have been neglected. The derivatives of the disturbance potential in (4.5) and (4.6) must be evaluated on the free surface, i.e. $z = \zeta + \eta$. This condition is met to each increasing order by expanding the disturbance potential in a Taylor series about $z = 0$, producing

$$\phi_z + \eta\phi_{zz} = \eta_t + J^{-1}[\Phi_\alpha \eta_\alpha + \phi_\alpha \zeta_\alpha + \Phi_\beta \eta_\beta + \phi_\beta \zeta_\beta],$$

$$\phi_t + \zeta\phi_{zt} + J^{-1}[\Phi_\alpha \phi_\alpha + \Phi_\alpha \zeta\phi_{z\alpha} + \Phi_\beta \phi_\beta] + \phi_z \phi_z + F^{-2}\eta = \text{constant},$$

where it is now understood that the derivatives of ϕ are evaluated at $z = 0$. These two conditions are combined by eliminating η to give

$$\begin{aligned} &J^3\phi_{tt} + 2J^2\Phi_\alpha\phi_{\alpha t} + 2\Phi_\beta\phi_{\beta t} + (\zeta_\alpha + \Phi_z)\phi_{zz} + (\Phi_{\alpha\alpha} - J_\alpha - F^{-2}\zeta_\alpha)\phi_\alpha \\ &+ (\Phi_{\alpha\beta} - F^{-2}\zeta_\beta)\phi_\beta + (J^3F^{-2} + \Phi_{\alpha z})\phi_z + J\Phi_\alpha^2\phi_{\alpha\alpha} + 2\Phi_\beta\phi_{\alpha\beta} + (\zeta_\alpha + \Phi_z)\phi_{\alpha z} \\ &+ F^{-2}\zeta\phi_{zz} + \zeta\phi_{ttz} + 2\zeta\phi_{\alpha tz} + \zeta\phi_{\alpha\alpha z} = 0. \end{aligned} \tag{4.7}$$

Equation (4.7) is the free-surface boundary condition for the disturbance potential, accurate to order a.

5. Instability

Using (4.7), it will be shown that the disturbance and the mean flow will interact, resulting in exponential growth of the disturbance with time, and indicating that the mean flow is unstable to some infinitesimal disturbances. The disturbance will be a pair of gravity waves of infinitesimal wave height satisfying the resonance conditions given by (1.1) and (1.2).

The primary waves studied here consist of an infinite sum of wave components that have crests that are not straight in (x, y)-space, but are straight in the (α, β)-space, and with wavenumbers that have a β-component as well as an α-component. To show that this oblique wave is unstable, the disturbance waves must also have a wavenumber that has both an α-component and a β-component. The variation in the α-direction for the disturbance potential (as yet unspecified) is chosen to be an exponential function. The mean flow must also be expressed in this way. Hence, the mean-flow equations for symmetric channels, (3.11)–(3.14) are equivalent to

$$\Phi = \alpha - \mathrm{i}a\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}k\alpha} - \mathrm{e}^{-\mathrm{i}k\alpha}) \sum_{n=0}^{\infty} B_n \cos n\pi\beta \cosh \gamma_n (z+d), \tag{5.1}$$

$$\zeta = -a\frac{1}{2k}(\mathrm{e}^{\mathrm{i}k\alpha} + \mathrm{e}^{-\mathrm{i}k\alpha}) \sum_{n=0}^{\infty} B_n \gamma_n \cos n\pi\beta \sinh \gamma_n d. \tag{5.2}$$

The Jacobian of the coordinate transformation is now

$$J = 1 + ak(\mathrm{e}^{\mathrm{i}k\alpha} + \mathrm{e}^{-\mathrm{i}k\alpha}) \cosh k\beta + O(a^2). \tag{5.3}$$

Similarly for antisymmetric channels:

$$\Phi = \alpha + \mathrm{i}a\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}k\alpha} - \mathrm{e}^{-\mathrm{i}k\alpha}) \sum_{n=1}^{\infty} B_n \sin (n-\tfrac{1}{2})\pi\beta \cosh \gamma_n (z+d), \tag{5.4}$$

$$\zeta = -a\frac{1}{2k}(\mathrm{e}^{\mathrm{i}k\alpha} + \mathrm{e}^{-\mathrm{i}k\alpha}) \sum_{n=1}^{\infty} B_n \gamma_n \sin (n-\tfrac{1}{2})\pi\beta \sinh \lambda_n d, \tag{5.5}$$

$$J = 1 + ak(\mathrm{e}^{\mathrm{i}k\alpha} + \mathrm{e}^{-\mathrm{i}k\alpha}) \sinh k\beta + O(a^2). \tag{5.6}$$

The disturbance waves are chosen to satisfy by themselves (2.2), (3.5), (3.6), (3.7) and (3.8) in their linear forms. There are two kinds of waves (in either a symmetric or antisymmetric channel) that can be found to satisfy these equations: symmetric about the $\beta = 0$ plane and antisymmetric about the $\beta = 0$ plane. The disturbance potentials for symmetric disturbances in a symmetric channel are chosen to be

$$\phi_m = \epsilon\, \mathrm{e}^{\mathrm{i}(m\alpha - \sigma t)} \cos L\pi\beta \cosh \gamma_L (z+d), \tag{5.7}$$

$$\phi_{m'} \epsilon'\, \mathrm{e}^{\mathrm{i}(m'\alpha - \sigma t)} \cos M\pi\beta \cosh \gamma_M (z+d). \tag{5.8}$$

The disturbance potentials for antisymmetric disturbances in a symmetric channel are

$$\phi_m = \epsilon\, \mathrm{e}^{\mathrm{i}(m\alpha - \delta t)} \sin (L-\tfrac{1}{2})\pi\beta \cosh \gamma_L (z+d), \tag{5.9}$$

$$\phi_{m'} = \epsilon'\, \mathrm{e}^{\mathrm{i}(m'\alpha - \sigma t)} \sin (M-\tfrac{1}{2})\pi\beta \cosh \gamma_M (z+d), \tag{5.10}$$

and the disturbances in an antisymmetric channel are

$$\phi_m = \epsilon\, \mathrm{e}^{\mathrm{i}(m\alpha - \sigma t)} \sin (L-\tfrac{1}{2})\pi\beta \cosh \gamma_L (z+d), \tag{5.11}$$

$$\phi_{m'} = \epsilon'\, \mathrm{e}^{\mathrm{i}(m'\alpha - \sigma t)} \cos M\pi\beta \cosh \gamma_M (z+d), \tag{5.12}$$

where L and M are any positive integers (except zero in (5.9), (5.10) and (5.11)). The use of L also for the channel half-width should not introduce confusion. The wave amplitudes, ϵ and ϵ', are assumed to be small enough so that their squares are negligible. The sum of any of the above pairs of potentials is the disturbance potential in (4.7), i.e.

$$\phi = \phi_m + \phi_{m'}. \tag{5.13}$$

Substituting ϕ_m and $\phi_{m'}$ separately into Laplace's equation produces

$$\left.\begin{aligned} \gamma_L^2 &= m^2 + L_i^2\pi^2, \\ \gamma_M^2 &= m'^2 + M^2 i\pi^2. \end{aligned}\right\} \tag{5.14}$$

where

$$L_i = \begin{cases} L, & L = 0,1,2,\ldots, \quad \text{if } i = 1, \\ L - \frac{1}{2}, & L = 1,2,3,\ldots, \quad \text{if } i = 2 \text{ or } 3, \end{cases}$$

$$M_i = \begin{cases} M, & M = 0,1,2,\ldots, \quad \text{if } i = 1 \text{ or } 3, \\ M - \frac{1}{2}, & M = 1,2,3,\ldots, \quad \text{if } i = 2, \end{cases}$$

where $i = 1$, 2 or 3 is valid for symmetric channels with symmetric disturbances, symmetric channels with antisymmetric disturbances, or antisymmetric channels with one symmetric disturbance and one antisymmetric disturbance, respectively.

The free-surface boundary condition, (4.7), gives†

$$\left.\begin{aligned} \lambda_0 &= m \pm [F^{-2}\gamma_L \tanh \gamma_L d]^{\frac{1}{2}}, \\ \lambda_0 &= m' \pm [F^{-2}\gamma_M \tanh \gamma_M d]^{\frac{1}{2}}, \end{aligned}\right\} \tag{5.15}$$

where the higher-order terms have been neglected and the free-surface condition is applied at $z = 0$, as for well-known infinitesimal waves.

The two disturbance wavetrains and the primary waves make up the wave triad which must satisfy the resonance condition for frequency, i.e. $\sigma_1 = \sigma_2 + \sigma_0$. The primary waves are stationary for which $\sigma_0 = 0$, hence the resonance condition becomes $\sigma_1 = \sigma_2$, which is shown in the disturbance potentials as σ. To demonstrate the instability of the mean flow, an expression for σ must be obtained when the mean flow and the disturbance waves exist simultaneously. This is accomplished by inserting the mean-flow potential, free-surface elevation for the mean flow, and the disturbance potential into the free-surface boundary condition, (4.7). After (4.7) has been expanded in this manner, the resulting terms are classified as free-wave terms or interaction terms (i.e. the term $J^3\phi_{tt}$ in (4.7) produces a free-wave term of order ϵ and interaction terms of order $a\epsilon$). The free-wave terms would be unchanged if there were no primary waves and the disturbance waves were being studied by themselves. The interaction terms are the nonlinear interactions between the mean flow and the disturbance waves. To show that the disturbance will grow with time, the interaction terms that have the same wavenumber vector as the free-wave terms are collected. It will be shown that the existence of such terms indicates that the mean flow and disturbances are interacting and exchanging energy.

The interaction terms that appear in the expansion of (4.7) have the form

$$e^{i(m\alpha - \sigma t)}\, e^{ik\alpha} \cos L\pi\beta \sum_{n=0}^{\infty} C_n \cos n\pi\beta, \tag{5.16}$$

or other similar forms. These terms will have the same wavenumber vector as the free-wave terms ((5.16) will produce terms like (5.8)) when

$$m' = m + k, \quad M = L + n. \tag{5.17\,a, b}$$

† The reason for the notation for σ, namely λ_0, will become apparent later in the discussion.

Bearing in mind that the frequencies of the disturbances have been assumed to be identical, (5.17) assures that the resonance conditions are satisfied.

For a chosen wall shape and disturbance combination, proceed by collecting the coefficients of terms in (4.7) which have the same wavenumber vector as ϕ_m, and then $\phi_{m'}$, producing two expressions which are given by McHugh (1986). Now σ is expanded as a power series in a:

$$\sigma = \lambda_0 + a\lambda_1 + O(a^2). \tag{5.18}$$

This expansion is inserted into (4.7) and the free-wave terms are subtracted, leaving two homogeneous expressions in ϵ and ϵ' which determine λ_1. The result is

$$\lambda_1^2 = \frac{G_1 G_2}{16(\lambda_0 - m)(\lambda_0 - m')}, \tag{5.16}$$

where

$$\begin{aligned} G_1 = {} & (\pm 1) M_i \pi [2\lambda_0 + k - 2m'] \Gamma_{1,i} - (\pm 1) 2k^{-1} F^{-2} M_i \pi \Gamma_{2,i} \\ & + [k^{-1} \gamma_M \tanh \gamma_M d \{-(\lambda_0 - m')^2 - k(2\lambda_0 + k - 2m')\} \\ & - m' F^{-2} + k^{-1} F^{-2} \gamma_M^2] \Gamma_{3,} - km' [2\lambda_0 + k - 2m'] \Gamma_{4,i} \\ & + [6\lambda_0^2 - 8\lambda_0 m' + 2km' + 2m'^2 - 6F^{-2} \gamma_M \tanh \gamma_M d] k \Gamma_{5,i}, \end{aligned}$$

$$\begin{aligned} G_2 = {} & (\pm 1) L_i \pi [2\lambda_0 - k - 2m] \Gamma_{1,i} - (\pm 1) 2k^{-1} F^{-2} L_i \pi \Gamma_{2,i} \\ & + [k^{-1} \gamma_L \tanh \gamma_L d \{-(\lambda_0 - m')^2 + k(2\lambda_0 - k - 2m)\} \\ & + mF^{-2} + k^{-1} F^{-2} \gamma_L^2] \Gamma_{3,i} - km(2\lambda_0 - k - 2m] \Gamma_{4,i} \\ & + [6\lambda_0^2 - 8\lambda_0 m - 2km + 2m^2 - 6F^{-2} \gamma_L \tanh \gamma_L d] \Gamma_{5,i}. \end{aligned}$$

In the above equations for G_1 and G_2 the interpretation of (± 1) is as follows: for $i = 1$, the positive is used, for $i = 2$, the negative is used, and for $i = 3$, the negative is used in G_1 and the positive in G_2.

The above expressions are valid for either symmetric or antisymmetric channels, depending on the value of the index i. For i = 1, the value of $\Gamma_{j,i}$ in (5.19) is defined to be

$$\Gamma_{j,i} (j = 1 \text{ or } 2) = \left\{ \begin{array}{ll} C_{L+M,j} \pm C_{|L-M|,j} & \text{if } L \neq 0, M \neq 0, \\ C_{L+M,j} & \text{if } L = M \neq 0, \\ C_{L,j} \text{ or } C_{M,j} & \text{if } L \neq 0, M = 0 \text{ or } L = 0, M \neq 0, \\ \text{None} & \text{if } L = M = 0, \end{array} \right\} \tag{5.20}$$

$$\Gamma_{j,i} (j = 3, 4 \text{ or } 5) = \left\{ \begin{array}{ll} C_{L+M,j} + C_{|L-m|,j} & \text{if } L \neq 0, M \neq 0, \\ 2C_{0,j} + C_{L+M,j} & \text{if } L = M \neq 0, \\ C_{L,j} \text{ or } C_{M,j} & \text{if } L \neq 0, M = 0 \text{ or } L = 0, M \neq 0, \\ 2C_{0,j} & \text{if } L = M = 0, \end{array} \right\} \tag{5.21}$$

where

$$\left. \begin{array}{ll} C_{n,1} = B_n n\pi \cosh \gamma_n d, & C_{n,2} = B_n \gamma_n n\pi \sinh \gamma_n d, \\ C_{n,3} = B_n \gamma_n \sinh \gamma_n d, & C_{n,4} = B_n \cosh \gamma_n d, \\ \gamma_n^2 C_{n,5} = 2k \sinh k (-1)^n. & \end{array} \right\} \tag{5.22}$$

For $i = 2$, $\Gamma_{j,i}$ are defined to be

$$\Gamma_{j,i} (j = 1 \text{ or } 2) = \left\{ \begin{array}{ll} C_{L,j} & \text{if } L = M, \\ \pm C_{|L-M|,j} + C_{L+M-1,j} & \text{if } L \neq M. \end{array} \right\} \tag{5.23}$$

$$\Gamma_{j,1} (j = 3, 4 \text{ or } 5) = \left\{ \begin{array}{ll} 2C_{0,j} - C_{L,j} & \text{if } L = M, \\ C_{|L-M|j,} - C_{L+M-1,j} & \text{if } L \neq M. \end{array} \right\} \tag{5.24}$$

The value of $C_{n,j}$ is defined again by (5.22). For $i = 3$, the value of $\Gamma_{j,i}$ is defined by (5.20) and (5.21), and $C_{n,j}$ are defined by (5.22), except for

$$\left.\begin{aligned} &C_{n,1} = B_n (n-\tfrac{1}{2})\pi \cosh \gamma_n d, \quad C_{n,2} = B_n \gamma_n (n-\tfrac{1}{2})\pi \sinh \gamma_n d, \\ &\gamma_n^2 C_{n,5} = 2k \cosh k(-1)^{n+1}. \end{aligned}\right\} \tag{5.25}$$

The mean flow is unstable if γ_1^2 is less than zero for any combination of L, M, m and m' that satisfies the resonance conditions.

6. Unstable modes

Phillips' resonance conditions must be satisfied for any interaction to occur. The frequency for the steady waves is zero, hence (1.1) results in

$$\sigma_1 = \sigma_2 = \sigma.$$

The first-order approximation for σ is given by (5.15). Solving for λ_0 in each of (5.15) and equating, then using (5.17 a), gives

$$\begin{aligned} &m \pm \{F^{-2}[m^2+L^2\pi^2]^{\frac{1}{2}} \tanh[m^2+L^2\pi^2]^{\frac{1}{2}} d\}^{\frac{1}{2}} \\ &\quad = (m+k) \pm \{F^{-2}[(m+k)^2+M^2\pi^2]^{\frac{1}{2}} \tanh[(m+k)^2+M^2\pi^2]^{\frac{1}{2}} d\}^{\frac{1}{2}}. \end{aligned} \tag{6.1}$$

The wavenumbers that demonstrate instability must satisfy (6.1). For a chosen set of F, k, d, L and M, the values of m that satisfy (6.1) can be found numerically. Using this value of m, corresponding values of m' and λ_0 can be found from (5.15), thereby completely defining the disturbance wavetrains.

Numerical values of m, λ_0 and λ_1^2, are given by McHugh (1986) for a non-dimensional depth d of unity and Froude numbers greater than unity. These results are similar to previous results found by Yih (1976) for flow over a wavy bottom. It is clear from the results of McHugh (1986) for $d = 1$ and $F > 1$, that the flow is always unstable, and the instability results from two disturbance waves that propagate downstream. Furthermore, there is only one pair of two-dimensional disturbance waves that can cause instability. There are, however, three-dimensional disturbances that can also cause instability in this range.

Now consider Froude numbers less than unity. Each of (5.15) can be expressed as

$$(\lambda_0 - m)^2 = F^{-2}\gamma \tanh \gamma d, \tag{6.2}$$

where $\gamma = m$ for two-dimensional disturbances. The left-hand side and the right-hand side of (6.2) are plotted versus m in figure 2 for a Froude number of 0.5 and a depth of unity. For the chosen value of λ_0, there are four intersections of the two curves in figure 2. Any pair of crossing points gives a set of the required disturbance wavetrains which satisfy (6.2); the value of k being the distance along the abscissa between the two crossing points. Hence the four intersections in figure 2 give six pairs of m–m' values. However, for a fixed value of k, there is a maximum of two pairs of disturbance wavetrains (type 1 and type 2) that satisfy the resonance conditions and could lead to an instability. Figure 3 is a plot of the values of m that satisfy the resonance conditions for a Froude number of 0.5 and values of k up to 10. The values of λ_0 corresponding to figure 3 are given in figure 4, and the resulting values of λ_1^2 are in figure 5.

The type-1 waves and the type-2 waves do not have a significant feature that easily distinguishes them from each other. Either type of wave, depending on the value of

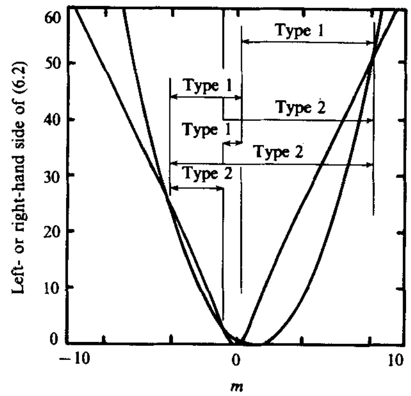

FIGURE 2. Left-hand side and right-hand side of (6.2) versus m for Froude number of 0.5.

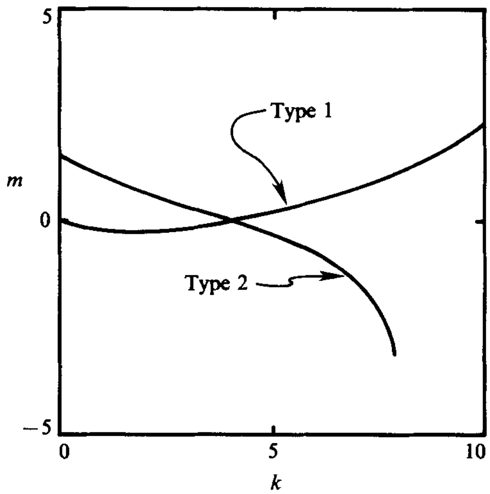

FIGURE 3. Disturbance wavenumbers for a Froude number of 0.5, a dimensionless depth of 1.0 and $L = M = 0$.

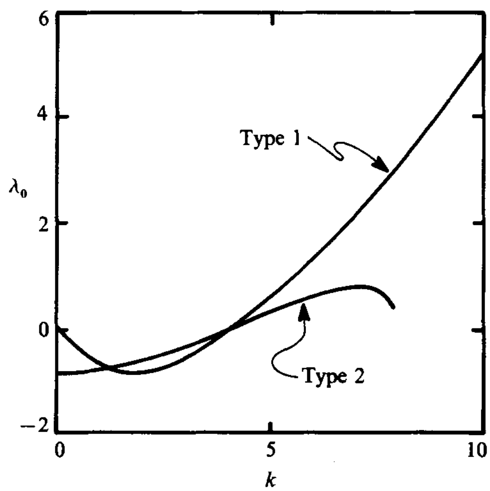

FIGURE 4. Frequency of disturbances for a Froude number of 0.5, a dimensionless depth of 1.0 and $L = M = 0$.

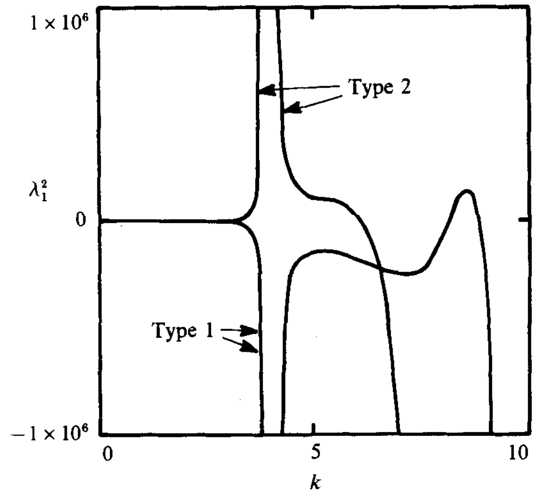

FIGURE 5. Values of λ_1^2 for a Froude number of 0.5, a dimensionless depth of 1.0 and $L = M = 0$.

k, can have sets of m–m' where m is negative and m' is positive, or vice versa. This means that one disturbance wave travels upstream and the other travels downstream. Both upstream- and downstream-propagating waves were seen by Binnie.

The type-1 waves give negative values of λ_1^2, as shown in figure 5, hence the flow is unstable to these disturbances. The value of λ_1^2 for the type-1 waves become positive for $7 < k < 8$ approximately. Since the type-2 waves do not exist for this range of k, it would appear that the flow is stable over this range of k. However this is not the case, and it will be demonstrated shortly that three-dimensional disturbances lead to instability for this range of k.

The value of λ_1^2 in figure 5 goes to infinity with k equal to 4 for both types of disturbances. This character is not attributable to the stability of the flow, but to the resonant condition of the primary waves. The denominator of (3.13), which gives the amplitude of the primary waves for values of k and F, contains C_n. C_n is zero for $k = 4$ and a Froude number of 0.5, resulting in an infinite value of B_0, which is the amplitude of the primary waves for $n = 0$. (The disturbances with $L = M = 0$ interact with the term in (3.11) with $n = 0$.) The infinite value of B_0 cause λ_1^2 to be infinite also. Hence the large absolute values of λ_1^2 near $k = 4$ may be artificial. Flow over a wavy bottom has a similar resonance, where the amplitude determined from the first-order approximation is infinite. W. Ellermeier (1983, personal communication) has shown for the wavy-bottom problem that when the perturbation theory is extended to the third order, the steady wave amplitude obtained is finite. This is likely to be true for the amplitude of the primary waves produced by the flow through the wavy-walled channels being studied here. However, it can be seen from figures 3 and 4 that a zero value of C_n results in a zero value for m and λ_0 in the calculation of the disturbance. Hence the resonant value for k is also a singular point for the resonance conditions.† Simmons (1969) has pointed out that such singular points result in secular behaviour of the stability solution. Therefore, if the stability analysis were carried out to the third order and the value of B_n was finite at the resonant value of k, as in Ellermeier, the solution for σ might still be secular owing to the singular behaviour of the resonance conditions.

† The author is indebted to a referee for pointing this out, and for the reference by Simmons.

Another difference between the results for wavy bottoms and wavy sides is for type-1 waves with $7 < k < 8$. For the wavy-bottom problem, the value of G (G is similar to λ_1^2) as given by Yih (1976) or McHugh (1986) decreases monotonically, while for the wavy-sided channel, the value of λ_1^2 is positive before going to negative infinity. This difference is due to the direct interaction between the disturbance waves and the wavy sidewalls, which does not occur for the wavy-bottom case. This interaction is represented in (4.7) by terms containing J, such as $J^3\phi_{tt}$, and in (5.19) by terms containing $C_{n,5}$. If these terms are removed from (5.19), the resulting values of λ_1^2 would decrease monotonically to infinity, as for the wavy-bottom case. Hence, the wavy sidewalls can be a stabilizing influence for certain flow conditions.

The values of m that satisfy the resonance conditions for various values of d, the dimensionless depth, are given by McHugh (1986) with corresponding values of λ_0 and λ_1^2. For a depth of 0.1, the value of λ_1^2 has been found to be positive for all k shown, indicating a stable flow. (Type-2 waves do not exist for a depth of 0.1.) The non-dimensional depth d is the dimensional depth divided by the half-width of the channel. Therefore, a small value of d could correspond to shallow water, or to a channel with the walls very far apart. The latter is significant since the positive values of λ_1^2 show that the flow through a wavy-walled channel can be stable if the walls are far enough apart. This explains how the bound waves near a curved wall in open water (such as the bound waves from a ship hull) can be stable while similar waves in a confined channel are not stable.

7. Three-dimensional disturbances

Consider now the more general case when both disturbances have non-zero wavenumbers in both the α- and β-direction. The flow can be unstable to such disturbances if a combination of L, M, m and m' can be found that satisfies (6.1) given the characteristics of the flow. Again (6.2) is used to show which types of disturbances exist, except now $\gamma^2 = p^2\pi^2 + m^2$, and p is a positive integer. Figure 6 contains a plot of the left-hand side of (6.2) and the right-hand side of (6.2) versus m for a series of values of p. The point where the left-hand side of (6.2) intersects any of the curves representing the right-hand side of (6.2) corresponds to a value of m and L or m' and M that satisfies (6.2). The distance along the abscissa between any two such intersections is the value of k that satisfies the resonance relations for those values of m, m', L and M. It is clear from figure 6 that the left-hand side intersects many of the curves of the right-hand side for a given value of k. However, for chosen non-zero values of L and M there are only two crossings between the curves in figure 6, representing one set of disturbance waves that satisfy the resonance conditions. With a value of $L = 0$ and $M \neq 0$, there are two types of waves, analogous to the two wave types for $L = M = 0$ in figure 2. Values of m for the type-1 waves are shown in figure 7 for $L = 0, 1, 2$ with $M = 0$, a Froude number of 0.5, and a depth of unity. Corresponding values of λ_0 and λ_1^2 are given in figures 8 and 9, respectively. Figure 9 shows that instability can be caused by three-dimensional disturbances.

The discussion of instability of flow through symmetric wavy walls to this point has concerned only symmetric disturbances. For antisymmetric disturbances, (6.1) is still valid, except now

$$\gamma_L^2 = (L-\tfrac{1}{2})^2\pi^2 + m^2, \quad \gamma_M^2 = (M-\tfrac{1}{2})^2\pi^2 + m'^2.$$

Since L and M cannot be zero for antisymmetric disturbances, these disturbances do not exist for the low range of k being considered.

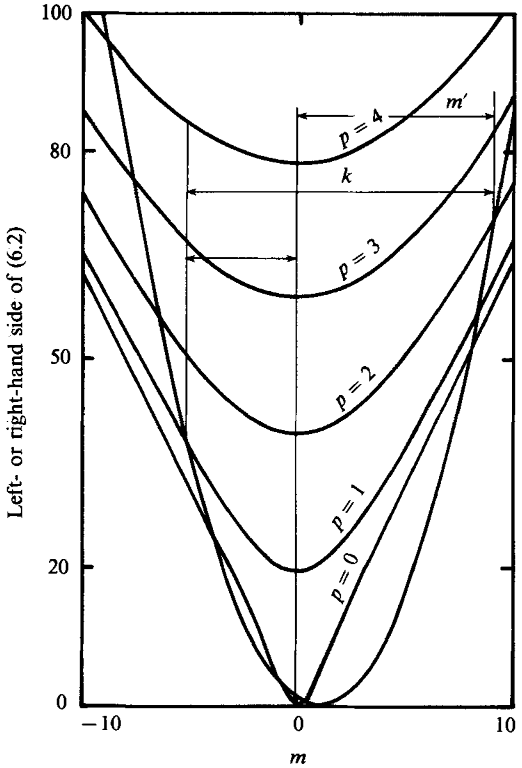

FIGURE 6. Left-hand side and right-hand side of (6.2) versus m for a Froude number less than unity and with L or M not equal to zero.

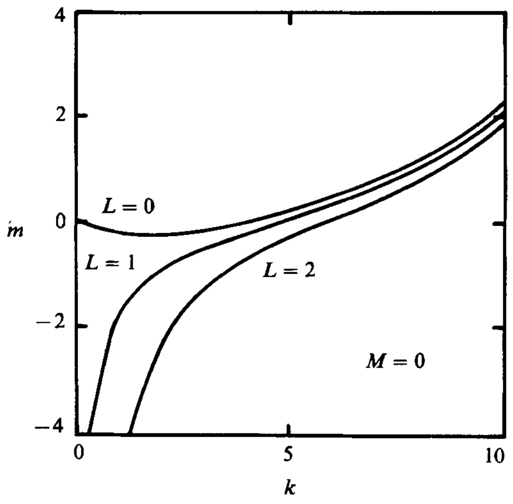

FIGURE 7. Disturbance wavenumbers for waves of type 1, a Froude number of 0.5, a dimensionless depth of 1.0, $L \neq 0$ and $M = 0$.

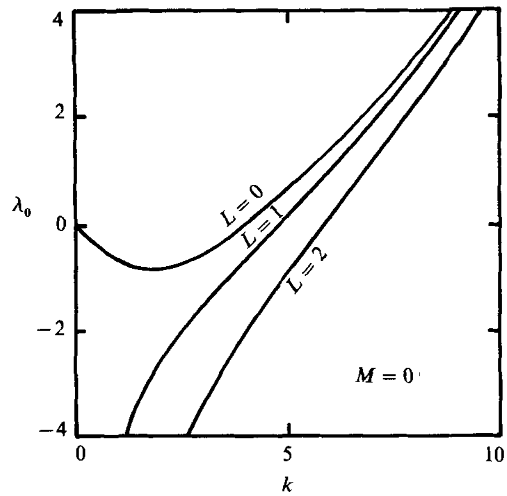

FIGURE 8. Frequency of type-1 disturbances for a Froude number of 0.5, a dimensionless depth of 1.0, $L \neq 0$ and $M = 0$.

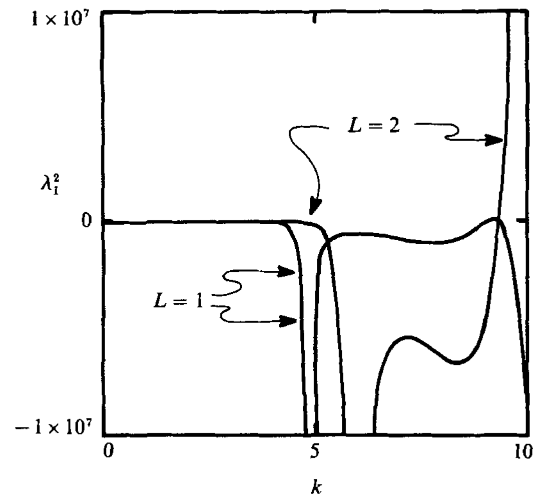

FIGURE 9. Values of λ_I^2 for type-1 waves with a Froude number of 0.5, a dimensionless depth of 1.0, $L \neq 0$ and $M = 0$.

8. Antisymmetric channels

Now antisymmetric channels will be considered briefly. The solution for the mean flow for antisymmetric channels given by (3.15) contains no term for $n = 0$. This means that every term in (3.15) has a dependence on β, and that at least one of the disturbances must have a dependence on β, i.e. at least one of the disturbances must be three-dimensional to satisfy the resonance conditions.

The instability in antisymmetric channels is similar to that in symmetric channels. There are, again, two types of disturbance waves that can cause instability for a chosen value of k. However, this is only true when the disturbance pair consists of one two-dimensional and one three-dimensional wave. Only one disturbance wave exists when both the disturbances are three-dimensional.

For the test conditions of Binnie, disturbances with an upstream wave and a downstream wave have been found, which lead to instability. Therefore, if Binnie's experiments were repeated in an antisymmetric channel, it is likely that a similar instability would again be witnessed.

9. Comparison with Binnie's experiments

Binnie measured or noted several distinctive features of the unsteady waves present in his experiments: (i) both an upstream and a downstream component were visible; (ii) the waves either had no transverse component of wavenumber or, more commonly, a transverse wavelength equal to the channel width; (iii) an accurate measure of the period; and (iv) an approximate measure of the longitudinal wavelength. Binnie used the transverse and longitudinal wavelengths to calculate a wave period, and found that the calculated period was close to the measured period (Binnie actually compared wave speed instead of period). The premise of this study is that the visible wavetrains of Binnie were one of a pair of disturbance wavetrains that satisfied the resonance conditions and were unstable. Considering this, the measured wavelengths and period of Binnie must define one of the two disturbance wavetrains. The question that must be answered is: What is the other disturbance wavetrain, and do the wavetrains result in a negative value of λ_1^2, indicating instability?

To answer this question, table 1*a* gives one of the test conditions of Binnie, along with associated dimensionless quantities defined previously. Using the basic channel and flow conditions of Binnie (i.e. k, F and d), an infinite number of disturbance wavetrains can be found that satisfy the resonance conditions, simply by allowing L or M to vary to infinity. The particular disturbance pair that is guilty of the instability in the experiments is the one with one of the disturbance waves matching, at least approximately, the measurements of Binnie. Table 1*b* contains disturbance pairs that closely match the test conditions of Binnie, and satisfy the resonance conditions. The frequency of each disturbance pair in table 1*b* is 5.86, matching the measured value in table 1*a*. The disturbances were found by choosing L and M, then allowing F to vary until the proper value for the frequency was achieved. This means that F and m do not match the experimental value exactly, and the disturbance that matches closest is likely to be the culprit. Only the disturbances that had a Froude number reasonably close to the measured value are shown. Note that there are no disturbances that exactly match all of measurements of Binnie. Disturbance (vi) in table 1(*b*) is quite close to all the parameters, with one exception. The values for m and m' are both positive, indicating both disturbance waves are propagating downstream. Binnie noted an upstream component as well.

Yih (1983) suggested that the instability found by Binnie was caused by a modulation in the channel shape, resulting in the walls containing wavelengths other than the ones listed. Furthermore, Yih noticed that the wavenumbers of Binnie were always an approximate multiple of the corrugation wavenumbers k, and suggested that the disturbance wavenumber m, was equal to k divided by an integer. To test this hypothesis of Yih, disturbances (viii)–(xii) are given in table 1(*b*) each satisfying the resonance conditions with a value of $k = 6.60$ (k/p where p is 2 for this particular experiment of Binnie). Disturbance (xii) again matches the measurements of Binnie quite well, and has an upstream and downstream component. However, the value of λ_1^2, which is proportional to the growth rate, for all the disturbances with a k of 6.6 is of order 10^6, while disturbances with a k of 13.19 (the actual k of the sidewalls) have (*a*) Binnie's measurements for one example case:

Channel width	12.6 in.
Wavelength	6 in.
Stream velocity	0.88 ft/s
Depth	8.40 in.
Period	0.64 s
Corrugation wavelength	3 in.
F	0.214
k	13.19
d	1.33
m	6.60
λ_0	5.86

(*b*) Disturbances which satisfy the resonance conditions and give $\lambda_0 = 5.86$:

	k	F	m	m'	L	M	λ_1^2
(i)	13.19	0.193	−12.17	1.02	0	0	0.135×10^{14}
(ii)	13.19	0.185	−15.41	−2.22	0	0	-0.275×10^{14}
(iii)	6.60	0.207	−5.52	1.08	0	0	0.547×10^{6}
(iv)	6.60	0.200	−9.99	33.39	0	0	-4.18×10^{6}
(v)	13.19	0.197	−12.17	1.02	1	0	-0.45×10^{14}
(vi)	13.19	0.187	−15.47	−2.28	1	0	0.51×10.14
(vii)	6.60	0.223	−5.43	1.17	1	0	-4.04×10^{6}
(viii)	6.60	0.203	−10.46	−3.86	1	0	0.27×10^{14}
(ix)	13.19	0.230	3.95	−12.14	1	7	-2.67×10^{14}
(x)	13.19	0.220	7.06	20.25	1	8	-2.66×10^{14}
(xi)	13.19	0.195	12.73	25.82	1	9	-2.34×10^{14}
(xii)	6.60	0.302	6.60	0	4	1	-14.17×10^{6}

TABLE 1. Comparison with the experiments of Binnie

a λ_1^2 of order 10^{14}. The disturbance with the larger growth rate would be expected to dominate.

Hence, a disturbance that matches all of Binnie's observations and has a dominant growth rate has not yet been found. It is possible that the instability was not a result of disturbances satisfying Phillips resonance conditions, but was in fact a result of some nonlinearity not yet discovered. The study of a viscous flow could prove to be particularly illuminating.

10. Conclusions

The following general conclusions can be drawn from the preceding discussion:

(i) The steady waves in both a symmetric and an antisymmetric channel are unstable, for any wavelength of the side corrugation, to disturbances that satisfy the Phillips resonance conditions.

(ii) For the Froude number, depth, and corrugation-wavelength ranges of the experiments of Binnie (1960), the steady waves in a symmetric channel are unstable to two disturbance wavetrains (which satisfy the Phillips resonance conditions), one of which propagates upstream and the other downstream. The steady waves in an antisymmetric channel are also unstable in the same way.

(iii) For all values of d, the depth over the half-length, the steady waves in a corrugated channel are stable. Hence the steady waves in a channel can be stabilized by moving the walls of the channel further apart, or by reducing the depth.

I want to thank Professor Chia-Shun Yih for suggesting this problem, and for many helpful discussions during the course of the work.

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